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On the Value of a Policy. By JAMES MEIKLE, *Actuary of the Scottish Provident Institution.*

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THE author commences by stating that he does not propose to consider what sums should in practice be given for the surrender of policies, or what is the value to a purchaser of a policy offered for sale in the public market; but on this occasion confines himself to the examination and explanation of the theoretical value of a policy.

He then points out that there are two methods of obtaining a formula for the value of a policy, the one of which is the simpler in practice, but the other gives a better understanding of the nature of that value, and better illustrates its progress from year to year.

He states, that in what follows the net premiums only are considered—the additions made to them in practice, or the loadings, being left out of consideration.

The substance of his introductory remarks is as follows:—

At the opening of the policy the value of the sum assured is exactly equal to the value of the net premiums; or, using the common notation,

$$\pi_x = \frac{A_x}{1+a_x}\ ;\ \text{and}\ A_x = (1+a_x)\pi_x;$$

or, taking a numerical example, the annual premium at the age 30 to insure £100 is 1·95192 (Carlisle 3 per cent.), and the value of an annuity of that amount payable in advance during the life of 30 is 1·95192 × 20·55694, or £40·1254, which is also the value of £100 payable on the death of the same person.

At the end of a year, supposing the life assured to be in existence at the beginning of the second year, the values of the obligations of the assured and the Company are altered as follows :—

The value of £100 payable on the death of a person now aged 31 is	£40·7304
The value of an annuity of £1·95192 in advance, on a life of 31 is 1·95192 × 20·34924 =	39·7200
Difference =	1·0104

This difference is the value of the policy at the end of the first year.

Proceeding in the same way, the differences at the end of ten successive years between the values of £100 payable at death and the values of the future premiums payable during life, will give the values of the policy at the end of those years respectively.

Value of £100	Payable at Death of	Value of future Premiums of £1·95192.	During Life of	Difference—being Value of Policy for £100 opened at Age 30.	At End of Years.
40·1254	30	40·1254	30	0	0
40·7304	31	39·7200	31	1·0104	1
41·3538	32	39·3022	32	2·0516	2
42·0069	33	38·8645	33	3·1424	3
42·6911	34	38·4060	34	4·2851	4
43·3971	35	37·9329	35	5·4642	5
44·1259	36	37·4445	36	6·6814	6
44·8679	37	36·9472	37	7·9207	7
45·6237	38	36·4407	38	9·1830	8
46·3938	39	35·9246	39	10·4692	9
47·1580	40	35·4125	40	11·7455	10

He then proceeds :—

The value of a policy thus consists of the difference between the value of the engagement of the Office to pay the sum assured, and the value of the engagement of the policy-holder to pay the stipulated premium during his life. This process of calculating the value of a policy illustrates one of the two views upon which it is proposed to dwell. It is based upon the variation of the engagements—the increasing value of the Office's liability for payment of the amount in the policy on death of a person of greater age,

and the diminishing value of the engagement to pay the premium during the shorter future lifetime of the person of a higher age. The elements of calculation have reference entirely to the future, and on that account, therefore, this method may be denoted the *prospective* method of calculating the value of a policy.

Employing $V_{x|n}$ to denote the value of a policy taken out at the age x, which has been n years in force, we shall have

$$V_{x|1}=A_{x+1}-\pi_x(1+a_{x+1})=\frac{M_{x+1}-\dfrac{M_x}{N_{x-1}}\cdot N_x}{D_{x+1}}$$

$$V_{x|2}=A_{x+2}-\pi_x(1+a_{x+2})=\frac{M_{x+2}-\dfrac{M_x}{N_{x-1}}\cdot N_{x+1}}{D_{x+2}}$$

&c. = &c.

If the premiums are terminable in m further payments, the value at the end of n years would be expressed by

$$A_{x+n}-\text{Premium}\times\{1+{}_{\overline{m-1}|}a_{x+n}\}$$

$$=\frac{M_{x+n}}{D_{x+n}}-\frac{M_x}{N_{x-1}-N_{x+n+m-1}}\times\frac{N_{x+n-1}-N_{x+n+m-1}}{D_{x+n}}$$

* * * *

Applying the same process to calculate the value of a policy for £100, which has been effected for a term of, say 10 years, on a life of 30, the premium for which is

$${}_{\overline{10}|}\pi_{30}=v-\frac{{}_{\overline{10}|}a_{30}}{1+{}_{\overline{9}|}a_{30}}=\frac{M_{30}-M_{40}}{N_{29}-N_{39}}=1\cdot0176 \text{ (Car. 3 per cent.)},$$

we get the results set forth in the following table :—

Value of £100 on Death of			Value of future Premiums of £1·0176 during			Difference—being Value of the Policy for £100	
Age	if he die within Years	£	Years	of Life.	£	At end of Years	£
30	10	8·561	10	30	8·561	0	·000
31	9	7·888	9	31	7·850	1	·038
32	8	7·179	8	32	7·110	2	·069
33	7	6·445	7	33	6·339	3	·106
34	6	5·690	6	34	5·537	4	·153
35	5	4·896	5	35	4·703	5	·193
36	4	4·058	4	36	3·835	6	·223
37	3	3·158	3	37	2·933	7	·225
38	2	2·190	2	38	1·994	8	·196
39	1	1·153	1	39	1·018	9	·135

Policies opened for short periods are thus shown to possess very small values. * * *

Applying again the same process to ascertain the value of an ordinary Endowment policy of £100 opened at age 30 and payable on attaining 40, for which the annual premium is £7·9548 :—

Value of £100 payable on reaching Age 40.		Value of future Premiums of £7·9548 payable till Age 40.		Difference—being Value of Policy.
	£		£	£
30	66·932	30	66·932	0
31	69·643	31	61·366	8·277*
32	72·472	32	55·581	16·891
33	75·410	33	49·557	25·853
34	78·461	34	43·285	35·176
35	81·644	35	36·763	44·881
36	84·965	36	29·980	54·985
37	88·447	37	22·928	65·519
38	92·100	38	15·592	76·508
39	95·934	39	7·955	87·979
40	100·	40	..	100·

And in the same way, the value of an *Endowment Assurance* policy effected at age 30 for £100, payable at death or on attaining age 40, for which the annual premium is £8·9724:—

Value of £100 at Death, or on attaining Age 40, to a Person who is now		Value of future Premiums of £8·9724, payable till Age 40.	Difference—being Value of Policy.
	£	£	£
30	75·493	75·493	0
31	77·531	69·216	8·315*
32	79·650	62·691	16·959
33	81·855	55·896	25·959
34	84·151	48·822	35·329
35	86·539	41·466	45·073
36	89·023	33·815	55·208
37	91·605	25·861	65·744
38	94·291	17·586	76·705
39	97·087	8·972	88·115
	100·	0·	100·

The foregoing views of the value of a policy refer exclusively to the prospective method of calculation, the general formula being

$$V_{x|n}=A_{x+n}-\pi_x(1+a_{x+n}).$$

* * * * *

Passing now to the *second* mode of valuing a policy, if we assume 5,642 persons (the number alive at age 30 in the Carlisle

* It will be observed that the difference between the values of the "endowment" policy and the "endowment assurance" policy, is exactly equal to the value of the "term assurance."—ED. *A. M.*

table of mortality), to assure their lives for £100 each, at a premium of £1·95192, the following will exhibit the progress of the fund:—

Year.	Number entering upon each Year.	Premiums received in each Year.	Interest received in each Year.	Claims paid in each Year.	Funds at end of each Year.
		£	£	£	£
1	5,642	11,012·7	330·3	5,700	5,643·0
2	5,585	10,901·4	496·3	5,700	11,340·8
3	5,528	10,790·2	663·9	5,600	17,194·9
4	5,472	10,680·9	836·3	5,500	23,212·1
5	5,417	10,573·5	1,013·6	5,500	29,299·2
6	5,362	10,466·2	1,192·9	5,500	35,458·3
7	5,307	10,358·8	1,374·5	5,600	41,591·6
8	5,251	10,249·5	1,555·2	5,700	47,696·4
9	5,194	10,138·2	1,735·0	5,800	53,769·6
10	5,136	10,025·1	1,913·8	6,100	59,608·6

From the foregoing it is manifest that the funds at the end of each of these years, are the accumulated surplus of premium paid by the persons assured, over and above the sum necessary to pay the claims calculated to become payable; and as these funds are the property of the members surviving at the end of each of these years, the share of each in these funds will be the value of his policy.*

Year.	Funds.	Divided by the Survivors.	Share to each, or Value of Policy.
	£		£
1	5,643·0	5,585	1·0104
2	11,340·8	5,528	2·0516
3	17,194·9	5,472	3·1424
4	23,212·1	5,417	4·2851
5	29,299·2	5,362	5·4642
6	35,458·3	5,307	6·6814
7	41,591·6	5,251	7·9207
8	47,696·4	5,194	9·1830
9	53,769·6	5,136	10·4692
10	59,608·6	5,075	11·7455

This mode of representing the value of a policy appears to me to be much more intelligible than the process, previously illustrated, of balancing the future engagements; and as it is confined entirely to the operations of the bygone years, I propose to name it the *retrospective* method of valuing a policy. It is represented by the following formulæ:—

* For further elucidation of this point, and generally on the subject of this paper, see Dr. Farr's article on the "Finance of Life Assurance," in the 12*th Report of the Registrar-General.*—Ed. *A. M.*

$$V_{x|n} = \pi_x \cdot \frac{1 + {}_{\overline{n-1}|}a_x}{p_{x|n}v^n} - \frac{v(1 + {}_{\overline{n-1}|}a_x) - {}_{\overline{n}|}a_x}{p_{x|n}v^n}$$

$$= \pi_x \cdot \frac{1 + {}_{\overline{n-1}|}a_x}{p_{x|n}v^n} - \left(v - \frac{{}_{\overline{n}|}a_x}{1 + {}_{\overline{n-1}|}a_x}\right)\frac{1 + {}_{\overline{n-1}|}a_x}{p_{x|n}v^n}$$

$$= (\pi_x - {}_{\overline{n}|}\pi_x)\frac{1 + {}_{\overline{n-1}|}a_x}{p_{x|n}v^n}$$

$$= \frac{\pi_x(N_{x-1} - N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$$

$$= \frac{(\pi_x + d)(N_{x-1} - N_{x+n-1}) - (D_x - D_{x+n})}{D_{x+n}}.$$

Formulæ which bear their own interpretation. We learn from them that the reason why policies have any value is that the progressively increasing risk of the assurance, from age to age, is paid for by an average premium, and as that average premium is always greater in the early years of the assurance than the risk in those years, the surplus part of the premium remains in hand, after payment of the risk, and gives rise to what has been termed the "value of the policy." The Office requires to husband that surplus part of the premium, and to make sure of accumulating it at the fundamental rate of interest, in order that it may provide for the years in which the risk is greater than the premium—the years of plenty must make up for the years of scarcity—and the sum of the values of policies at an investigation, compared with the funds actually in hand, is the mode by which that stewardship is put to the test. If the policy-holder merely paid the value of each year's risk, the policy would possess *no* value; and whatever sum he pays more than the average premium, would form a further increasing element in the value of the policy. If the policy-holder paid a premium without the Office incurring any risk, as in the case of endowments, the value of the policy would be the accumulated amount of the premiums paid by all the assured, divided amongst those who survived.

These statements will be more apparent from the following views of the progress of the fund for such transactions:—

1. *Temporary* assurance. ${}_{\overline{10}|}\pi_{30} = 1\cdot0176$.

Year.	Number entering upon each Year.	Premiums received in each Year.	Interest received each Year.	Claims paid in each Year.	Funds at end of each Year.	Number surviving each Year.	Share to each, or Value of Policy.
		£	£	£	£		£
1	5,642	5,741·17	172·23	5,700	213·40	5,585	·038
2	5,585	5,683·17	176·90	5,700	373·47	5,528	·068
3	5,528	5,625·16	179·96	5,600	578·59	5,472	·106
4	5,472	5,568·18	184·40	5,500	831·17	5,417	·153
5	5,417	5,512·21	190·30	5,500	1,033·68	5,362	·193
6	5,362	5,456·25	194·70	5,500	1,184·63	5,307	·223
7	5,307	5,400·28	197·55	5,600	1,182·46	5,251	·225
8	5,251	5,343·30	195·77	5,700	1,021·53	5,194	·197
9	5,194	5,285·30	189·20	5,800	696·03	5,136	·136
10	5,136	5,226·28	177·67	6,100	..	5,075	

These values of the policies are the same as those brought out by the former method. As the yearly premium, when the assurance is for a short term of years, very slightly exceeds the risk, these policies possess very small values.

2. *Endowment.* Here the premium for an endowment of £100 on 30, payable at 40, is 7·9548.

Year.	Number entering each Year.	Premiums paid each Year.	Interest received each Year.	Claims.	Funds at end of each Year.	Number surviving each Year.	Share to each, or Value of Policy.
		£	£		£		£
1	5,642	44,881·2	1,346·4	..	46,227·6	5,585	8·277
2	5,585	44,427·8	2,719·7	..	93,375·1	5,528	16·891
3	5,528	43,974·3	4,120·5	..	141,469·9	5,472	25·853
4	5,472	43,528·9	5,550·0	..	190,548·8	5,417	35·177
5	5,417	43,091·3	7,009·2	..	240,649·3	5,362	44·880
6	5,362	42,653·8	8,499·1	..	291,802·2	5,307	54·984
7	5,307	42,216·3	10,020·5	..	344,039·0	5,251	65·519
8	5,251	41,770·8	11,574·3	..	397,384·1	5,194	76·507
9	5,194	41,317·4	13,161·0	..	451,862·5	5,136	87·979
10	5,136	40,856·1	14,781·4	..	507,500·0	5,075	100·000

In this case the value of the policy is seen to consist of the premiums paid by the assured accumulated at 3 per cent. interest, which is further increased by the share of the premiums paid by those who have died in the previous years.

The formula is,

$$\text{Value after } n \text{ years} = \text{Premium} \times \frac{1+{}_{\overline{n-1}|}a_x}{p_{x|n}v^x}$$

$$= \text{Premium} \times \frac{N_{x-1}-N_{x+n-1}}{D_{x+n}}.$$

If there were any number of endowments opened at the same

age for different sums, payable at different ages, but covered by the same amount of premium, their values would be the same.

Thus—

Endowment of £19·109 to 30 at 40	by an annual premium of	£1·52
and of £49·158 to 30 at 50	„ „	£1·52
and of £100· to 30 at 60	„ „	£1·52

their values would be—

1st year=£1·582		7th year=£12·520	
2nd „	3·228	8th „	14·620
3rd „	4·940	9th „	17·204
4th „	6·722	10th „	19·109
5th „	8·576	11th „	21·528
6th „	10·507	12th „	24·071

3. *Endowment assurance.* The premium for an endowment assurance of £100 on 30, payable at 40 or on previous death, is 8·9724.

Year.	Number entering upon each Year.	Premiums received each Year.	Interest each Year.	Claims paid each Year.	Funds at end of each Year.	Number surviving each Year.	Share to each, or Value of Policy.
		£	£	£	£		£
1	5,642	50,622	1,519	5,700	46,441	5,585	8·315
2	5,585	50,111	2,897	5,700	93,748	5,528	16·959
3	5,528	49,599	4,300	5,600	142,048	5,472	25·959
4	5,472	49,097	5,734	5,500	191,380	5,417	35·329
5	5,417	48,604	7,199	5,500	241,683	5,362	45·073
6	5,362	48,110	8,694	5,500	292,987	5,307	55·207
7	5,307	47,617	10,218	5,600	345,221	5,251	65·744
8	5,251	47,114	11,770	5,700	398,406	5,194	76·705
9	5,194	46,603	13,350	5,800	452,558	5,136	88·115
10	5,136	46,082	14,959	6,100	507,500	5,075	100·

In this case, the value of the policy is the share to the survivor of the difference between the accumulated premiums and accumulated claims.

$$\text{Value after } n \text{ years} = \text{Premium} \times \frac{1+{}_{\overline{n-1}|}a_x}{p_{x|n}v^n} - \frac{v(1+{}_{\overline{n-1}|}a_x)-{}_{\overline{n}|}a_x}{p_{x|n}v^n}$$

$$= \text{Premium} \times \frac{N_{x-1}-N_{x+n-1}}{D_{x+n}} - \frac{M_x - M_{x+n}}{D_{x+n}}.$$

This formula may be very much abridged, as shall be afterwards pointed out.

Such are the two views of the value of a policy which I have desired to lay before you—the prospective and the retrospective.

There may be nothing new about them; both formulæ may be very well known; but we may learn something new by their comparison.

* * * * *

Passing from the explanatory and the illustrative departments of my subject to the analytical and the practical, the previous formulæ may be put into some very neat and elegant forms. Thus—

$$
\begin{aligned}
V_{x|n} &= A_{x+n} - \pi_x(1+a_{x+n}) \\
&= v(1+a_{x+n}) - a_{x+n} - \frac{v(1+a_x)-a_x}{1+a_x}(1+a_{x+n}) \\
&= \frac{v(1+a_{x+n})(1+a_x) - a_{x+n} - a_x\,a_{x+n} - v(1+a_x)(1+a_{x+n}) + a_x + a_x a_{x+n}}{1+a_x} \\
&= \frac{a_x - a_{x+n}}{1+a_x} = 1 - \frac{1+a_{x+n}}{1+a_x}.
\end{aligned}
$$

This formula is applicable to the case of joint lives,

$$V_{x.y|n} = 1 - \frac{1+a_{x+n.y+n}}{1+a_{x.y}};$$

and to the survivor of two lives,

$$1 - \frac{1+a_{\overline{x+n.y+n}}}{1+a_{\overline{x.y}}} = 1 - \frac{1+a_{x+n}+a_{y+n}-a_{x+n.y+n}}{1+a_x+a_y-a_{x.y}}.$$

It is a very valuable formula in the case of whole life assurance on single lives, not so much for the abbreviation of the calculation, as from the risk of error being in a great measure removed. If, in the formula $V_{x|n} = A_{x+n} - \pi_x(1+a_{x+n})$, any error were contained in π_x, the resulting value would contain that error multiplied many times; and if the same error is introduced into the other formula for $V_{x|n} = \frac{\pi_x(N_{x-1}-N_{x+n-1}) - (M_x - M_{x+n})}{D_{x+n}}$, it will be observed that the resulting value would be affected in a different direction. * * * *

This formula for the value of a policy,

$$V_{x|n} = 1 - \frac{1+a_{x+n}}{1+a_x},$$

is the one usually followed in the calculation of tables of the values of policies; the principal thing required being the logarithm of the annuities in advance. * * * *

Recently I have adopted a check for these and similar calculations, by the process of summation:—

$$V_{x|1}+V_{x|2}+\&c.\ldots V_{x|n}=\frac{na_x-(a_{x+1}+a_{x+2}+\ldots a_{x+n})}{1+a_x},$$

and forming a table of the sum of

$$a_z+a_{z-1}+\&c.\ a_x=\Sigma a_x$$

$$\Sigma V_{x|n}=\frac{na_x-(\Sigma a_{x+1}-\Sigma a_{x+n+1})}{1+a_x}.$$

The formula for the value of a temporary assurance policy is not capable of any very elegant expression in terms of the fundamental annuities. The value of an endowment assurance policy, however, is capable of being compressed into a similarly elegant expression as that of a whole life premium policy.

Value at end of n years of an endowment assurance on x, payable at $x+m$ or previous death,

$$=\frac{{}_{\overline{m-1}|}a_x-{}_{\overline{m-n-1}|}a_{x+n}}{1+{}_{\overline{m-1}|}a_x}$$

$$=1-\frac{1+{}_{\overline{m-n-1}|}a_{x+n}}{1+{}_{\overline{m-1}|}a_x}$$

$$=1-\frac{N_{x+n-1}-N_{n+m-1}}{N_{x-1}-N_{x+m-1}}\cdot\frac{D_x}{D_{x+n}}.$$

The value of an endowment policy has been shown to be respectively

$$p_{x+n|m-n}v^{m-n}-\text{Premium}\times(1+{}_{\overline{m-n-1}|}a_{x+n}),$$

$$\text{and Premium}\times\frac{1+{}_{\overline{n-1}|}a_x}{p_{x|n}v^n},$$

which can be shown to be equal to each other.

The paper concludes with a proof that the value of an ordinary policy may be expressed by any one of the following formulæ :—

(1) $$\frac{\pi_{x+n}-\pi_x}{\pi_{x+n}+d}.$$

(2) $$(\pi_{x+n}-\pi_x)(1+a_{x+n}).$$

(3) $$1-(d+\pi_x)(1+a_{x+n}).$$

(4) $$\frac{1}{1+\frac{1}{(1+a_x)(\pi_{x+n}-\pi_x)}}.$$

(5) $$1-\frac{1-A_{x+n}}{1-A_x}.$$

We have also the following relation — more curious than useful :—

$$V_{x+n|m-n}=1-\frac{1-V_{x|m}}{1-V_{x|n}}.$$

Again,

$$1-V_{x|n}=\frac{1+a_{x+n}}{1+a_x}=\frac{\pi_x+d}{\pi_{x+n}+d}=\frac{1-A_{x+n}}{1-A_x},$$

formulæ which are very useful in many actuarial investigations.

Determination and Distribution of Profit. *By* James Meikle, *Actuary of the Scottish Provident Institution.*

[Read before the Institute, 29th February, 1864, and printed in abstract by order of the Council.]

In the paper which I had recently the pleasure of submitting to the Institute, I gave an explanation of what was meant by the phrase "the value of a policy." It was therein stated that, *in one view*, the value of a policy was the difference between the present value of the sum assured payable by the Office, and the present value of the premiums payable by the person assured; and *in another view*, that the value of a policy was the difference between the premiums received and the claims paid, accumulated at the rates of mortality and interest inherent in the calculation of the premiums. The *first* of these being a view of the value in respect of the *future* years; and the *second*, in respect of the *past* years. It was further stated, that so long as the experienced rates of mortality and interest coincided with the expected rates, the results brought out by these two methods of calculation would be the same. I shall now consider values of policies when the rates of mortality and interest which have been experienced are *different* from those inherent in the original calculations—that is to say, when different rates have been experienced in the past years from the rates which it was expected would have been experienced, or which it is expected will be experienced in the future years. I propose, in short, to consider the subject of profit; to trace its sources, and to state the modes in which it should be determined and distributed among the policy-holders.

The author remarks, that "few actuaries have written upon profit, the sources of its creation, and the modes of its distribution," and that those who have done so have not been sufficiently elemen-